THE DIFFERENTIAL EQUATION OF THE THIRD ORDER WITH A QUADRATIC RELATION BETWEEN THE INTEGRALS

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Let

$$T(y) \equiv \frac{d^3y}{dx^3} + t_1(x)\frac{d^2y}{dx^2} + t_2(x)\frac{dy}{dx} + t_3(x)y = 0$$

be a linear homogeneous differential equation with rational coefficients, and

$$y_1, \quad y_2, \quad y_3$$

a fundamental system of integrals between which there is a quadratic relation

$$y_2^2 - y_1y_3 = 0. \tag{1}$$

Fuchs has shown* that there exists a linear homogeneous differential equation of second order, with rational coefficients,

$$S(z) \equiv \frac{d^2z}{dx^2} + s_1(x)\frac{dz}{dx} + s_2(x)z = 0,$$

such that

$$y_1 = z_1^2, \quad y_2 = z_1z_2, \quad y_3 = z_2^2,$$

the functions z_1 and z_2 forming a fundamental system of $S = 0$.

Vessiot† has considered the equation of third order $T(y) = 0$ with the relation between the integrals

$$y_2^2 - y_1y_3 = r(x), \tag{2}$$

where $r(x)$ is a rational function of x.

It will be shown in this note that *by the transformation* (4) *the equation* $T(y) = 0$ *can be transformed to an equation of third order* $\overline{T}(\overline{y}) = 0$ *such that*

* L. Fuchs, *Acta Mathematica*, vol. 1, 1882, pp. 321–362; Picard, *Traité d'Analyse*, vol. 3, (1908), pp. 585–588.

† E. Vessiot, *Annales de l'Ecole Normale Supérieure*, vol. 9, (1892), pp. 274–180.

$$\bar{y}_2^2 - \bar{y}_1\bar{y}_3 = 0. \tag{3}$$

This proves that the discussion of $T = 0$ with the relation (2) between its integrals is equivalent to Fuchs' (seemingly more special) problem of discussing an equation of third order with the relation (1) between its integrals.

The equation

$$\bar{y} = p_0 y + p_1 y' + p_2 y'', \tag{4}$$

the coefficients p_0, p_1, p_2 denoting (for the present undetermined) functions of x, can, in general, be solved for y. By differentiating twice, and making use of the fact that y is an integral of $T = 0$, we obtain

$$\bar{y}' = (p_0' - p_2 t_3)y + (p_0 + p_1' - p_2 t_2)y' + (p_1 + p_2' - p_2 t_1)y'', \tag{4'}$$

$$\begin{aligned} \bar{y}'' = {} & [(p_0' - p_2 t_3)' - t_3(p_1 + p_2' - p_2 t_1)]y \\ & + [(p_0' - p_2 t_3) + (p_0 + p_1' - p_2 t_2)' - t_2(p_1 + p_2' - p_2 t_1)]y' \\ & + [(p_0 + p_1' - p_2 t_2) + (p_1 + p_2' - p_2 t_1)' - t_1(p_1 + p_2' - p_2 t_1)]y''. \end{aligned} \tag{4''}$$

The solution gives

$$y = q_0\bar{y} + q_1\bar{y}' + q_2\bar{y}'', \tag{5}$$

where q_0, q_1, q_2 are functions of x, provided we exclude values of p_0, p_1, p_2 (and consequently of q_0, q_1, q_2), called *singular*, which lead to a vanishing determinant of the coefficients on the right hand side of (4), (4′), (4″).

It is well known* that $\bar{y}$ also satisfies a linear homogeneous differential equation of third order

$$\bar{T}(\bar{y}) \equiv \frac{d^3\bar{y}}{dx^3} + \bar{t}_1(x)\frac{d^2\bar{y}}{dx^2} + \bar{t}_2(x)\frac{d\bar{y}}{dx} + \bar{t}_3(x)\bar{y} = 0.$$

We proceed to show that there exists a fundamental system $\bar{y}_1, \bar{y}_2, \bar{y}_3$ of $\bar{T} = 0$ such that

$$\bar{y}_2^2 - \bar{y}_1\bar{y}_3 = 0, \tag{3}$$

while between the integrals of $T = 0$ the relation is

* L. Schlesinger, *Handbuch der Theorie der linearen Differentialgleichungen*, vol. 2, part 1, p. 114.

(2) $$y_2^2 - y_1y_3 = r(x).$$

Assuming the equality (3) we have from (4)

$$(p_0y_2 + p_1y_2' + p_2y_2'')^2 - (p_0y_1 + p_1y_1' + p_2y_1'')(p_0y_3 + p_1y_3' + p_2y_3'') = 0,$$

or, collecting,

(6) $$p_0^2(y_2^2 - y_1y_3) + p_1^2(y_2'^2 - y_1'y_3') + p_2^2(y_2''^2 - y_1''y_3'') + p_0p_1(2y_2y_2' - y_1y_3' - y_1'y_3) + p_0p_2(2y_2y_2'' - y_1y_3'' - y_1''y_3) + p_1p_2(2y_2'y_2'' - y_1'y_3'' - y_1''y_3') = 0.$$

By differentiating the equation (2) five times and eliminating derivatives beyond the second by means of $T(y) = 0$, we obtain five relations with rational coefficients which are linear in the following five functions:—

$$y_2'^2 - y_1'y_3' = r_1(x), \qquad y_2''^2 - y_1''y_3'' = r_2(x), \qquad 2y_2y_2' - (y_1y_3' + y_1'y_3) = r_3(x)$$

$$2y_2y_2'' - (y_1y_3'' + y_1''y_3) = r_4(x), \qquad 2y_2'y_2'' - (y_1'y_3'' + y_1''y_3') = r_5(x);$$

whence it appears that these five functions are rational in x.

Substituting these values in (6), there results

(7) $$p_0^2r + p_1^2r_1 + p_2^2r_2 + p_0p_1r_3 + p_0p_2r_4 + p_1p_2r_5 = 0.$$

In general, we may let

$$p_1 = 0, \qquad p_2 = 1$$

and obtain the equation

(7′) $$rp_0^2 + r_4p_0 + r_2 = 0$$

for the determination of p_0. Thus the transformation

(4′) $$\bar{y} = p_0y + y''$$

takes $T(y) = 0$ into $\bar{T}(\bar{y}) = 0$ and $y_2^2 - y_1y_3 = r(x)$ into $\bar{y}_2^2 - \bar{y}_1\bar{y}_3 = 0$. In case it happens that

$$p_0 = \frac{-r_4 \pm \sqrt{r_4^2 - 4rr_2}}{2r}, \qquad p_1 = 0, \qquad p_2 = 1$$

are singular values of the p's, then some other arbitrary choice of p_1 and p_2 together with the value of p_0 computed from (7) should be taken.*

* The domain of rationality of this investigation consists of: (1) all complex numbers, (2) the coefficients t_1, t_2, t_3 of the equation $T = 0$, the function $r(x)$, and (3) the operations of addition, subtraction, multiplication, division (exclusive of the null divisor), extraction of square roots, and differentiation.

It remains still to be shown that $\overline{y}_1, \overline{y}_2, \overline{y}_3$ form a fundamental system of $\overline{T}(\overline{y}) = 0$, where

$$y_i = q_0\overline{y}_i + q_1\overline{y}_i' + q_2\overline{y}_i'' \qquad (i = 1, 2, 3). \tag{5'}$$

If these functions are not linearly independent, we will have

$$c_1\overline{y}_1 + c_2\overline{y}_2 + c_3\overline{y}_3 = 0\,;$$

from this equation and (5′) it follows that

$$c_1y_1 + c_2y_2 + c_3y_3 = 0,$$

a condition which is impossible, since y_1, y_2, y_3 are a fundamental system of $T(y) = 0$.

Note. The propriety of replacing the equation $T(y) = 0$ by $\overline{T}(\overline{y}) = 0$ may be made clear from at least two points of view. I. If $\overline{T} = 0$ is integrated then, by (5) a fundamental system of $T = 0$ is known at once; and inversely, if $T = 0$ is integrated, then a fundamental system of $\overline{T} = 0$ is known by (4). II. From the standpoint of the Picard-Vessiot Theory, the equations $T = 0$ and $\overline{T} = 0$, being cogredient,* have the same Rationality Group, and consequently, the same integration theory.

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* Schlesinger, loc. cit. pp. 115 and 121.