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# THE DIFFERENTIAL EQUATION OF THE THIRD ORDER WITH A QUADRATIC RELATION BETWEEN THE INTEGRALS

BY S. EPSTEEN

LET

$$T(y) \equiv \frac{d^3y}{dx^3} + t_1(x) \frac{d^2y}{dx^2} + t_2(x) \frac{dy}{dx} + t_3(x)y = 0$$

be a linear homogeneous differential equation with rational coefficients, and

$$y_1, y_2, y_3$$

a fundamental system of integrals between which there is a quadratic relation

$$(1) \quad y_2^2 - y_1y_3 = 0.$$

Fuchs has shown\* that there exists a linear homogeneous differential equation of second order, with rational coefficients,

$$S(z) \equiv \frac{d^2z}{dx^2} + s_1(x) \frac{dz}{dx} + s_2(x)z = 0,$$

such that

$$y_1 = z_1^2, \quad y_2 = z_1z_2, \quad y_3 = z_2^2,$$

the functions  $z_1$  and  $z_2$  forming a fundamental system of  $S = 0$ .

Vessiot† has considered the equation of third order  $T(y) = 0$  with the relation between the integrals

$$(2) \quad y_2^2 - y_1y_3 = r(x),$$

where  $r(x)$  is a rational function of  $x$ .

It will be shown in this note that by the transformation (4) the equation  $T(y) = 0$  can be transformed to an equation of third order  $\bar{T}(\bar{y}) = 0$  such that

\* L. Fuchs, *Acta Mathematica*, vol. 1, 1882, pp. 321-362; Picard, *Traité d'Analyse*, vol. 3, (1908), pp. 585-588.

† E. Vessiot, *Annales de l'Ecole Normale Supérieure*, vol. 9, (1892), pp. 274-180.

$$(3) \quad \bar{y}_2^2 - \bar{y}_1\bar{y}_3 = 0.$$

This proves that the discussion of  $T = 0$  with the relation (2) between its integrals is equivalent to Fuchs' (seemingly more special) problem of discussing an equation of third order with the relation (1) between its integrals.

The equation

$$(4) \quad \bar{y} = p_0y + p_1y' + p_2y'',$$

the coefficients  $p_0, p_1, p_2$  denoting (for the present undetermined) functions of  $x$ , can, in general, be solved for  $y$ . By differentiating twice, and making use of the fact that  $y$  is an integral of  $T = 0$ , we obtain

$$(4') \quad \bar{y}' = (p_0' - p_2t_3)y + (p_0 + p_1' - p_2t_2)y' + (p_1 + p_2' - p_2t_1)y'',$$

$$(4'') \quad \begin{aligned} \bar{y}'' = & [(p_0' - p_2t_3)' - t_3(p_1 + p_2' - p_2t_1)]y \\ & + [(p_0' - p_2t_3) + (p_0 + p_1' - p_2t_2)' - t_2(p_1 + p_2' - p_2t_1)]y' \\ & + [(p_0 + p_1' - p_2t_2) + (p_1 + p_2' - p_2t_1)' - t_1(p_1 + p_2' - p_2t_1)]y''. \end{aligned}$$

The solution gives

$$(5) \quad y = q_0\bar{y} + q_1\bar{y}' + q_2\bar{y}'',$$

where  $q_0, q_1, q_2$  are functions of  $x$ , provided we exclude values of  $p_0, p_1, p_2$  (and consequently of  $q_0, q_1, q_2$ ), called *singular*, which lead to a vanishing determinant of the coefficients on the right hand side of (4), (4'), (4'').

It is well known\* that  $\bar{y}$  also satisfies a linear homogeneous differential equation of third order

$$\bar{T}(\bar{y}) \equiv \frac{d^3\bar{y}}{dx^3} + \bar{t}_1(x) \frac{d^2\bar{y}}{dx^2} + \bar{t}_2(x) \frac{d\bar{y}}{dx} + \bar{t}_3(x)\bar{y} = 0.$$

We proceed to show that there exists a fundamental system  $\bar{y}_1, \bar{y}_2, \bar{y}_3$  of  $\bar{T} = 0$  such that

$$(3) \quad \bar{y}_2^2 - \bar{y}_1\bar{y}_3 = 0,$$

while between the integrals of  $T = 0$  the relation is

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\* L. Schlesinger, *Handbuch der Theorie der linearen Differentialgleichungen*, vol. 2, part 1, p. 114.

$$(2) \quad y_2^2 - y_1 y_3 = r(x).$$

Assuming the equality (3) we have from (4)

$$(p_0 y_2 + p_1 y_2' + p_2 y_2'')^2 - (p_0 y_1 + p_1 y_1' + p_2 y_1'')(p_0 y_3 + p_1 y_3' + p_2 y_3'') = 0,$$

or, collecting,

$$(6) \quad p_0^2 (y_2^2 - y_1 y_3) + p_1^2 (y_2'^2 - y_1' y_3') + p_2^2 (y_2''^2 - y_1'' y_3'') + p_0 p_1 (2y_2 y_2' - y_1 y_3' - y_1' y_3) \\ + p_0 p_2 (2y_2 y_2'' - y_1 y_3'' - y_1'' y_3) + p_1 p_2 (2y_2' y_2'' - y_1' y_3'' - y_1'' y_3') = 0.$$

By differentiating the equation (2) five times and eliminating derivatives beyond the second by means of  $T(y) = 0$ , we obtain five relations with rational coefficients which are linear in the following five functions:—

$$y_2^2 - y_1 y_3 = r_1(x), \quad y_2'^2 - y_1' y_3' = r_2(x), \quad 2y_2 y_2' - (y_1 y_3' + y_1' y_3) = r_3(x) \\ 2y_2 y_2'' - (y_1 y_3'' + y_1'' y_3) = r_4(x), \quad 2y_2' y_2'' - (y_1' y_3'' + y_1'' y_3') = r_5(x);$$

whence it appears that these five functions are rational in  $x$ .

Substituting these values in (6), there results

$$(7) \quad p_0^2 r + p_1^2 r_1 + p_2^2 r_2 + p_0 p_1 r_3 + p_0 p_2 r_4 + p_1 p_2 r_5 = 0.$$

In general, we may let

$$p_1 = 0, \quad p_2 = 1$$

and obtain the equation

$$(7') \quad r p_0^2 + r_4 p_0 + r_2 = 0$$

for the determination of  $p_0$ . Thus the transformation

$$(4') \quad \bar{y} = p_0 y + y''$$

takes  $T(y) = 0$  into  $\bar{T}(\bar{y}) = 0$  and  $y_2^2 - y_1 y_3 = r(x)$  into  $\bar{y}_2^2 - \bar{y}_1 \bar{y}_3 = 0$ . In case it happens that

$$p_0 = \frac{-r_4 \pm \sqrt{r_4^2 - 4r r_2}}{2r}, \quad p_1 = 0, \quad p_2 = 1$$

are singular values of the  $p$ 's, then some other arbitrary choice of  $p_1$  and  $p_2$  together with the value of  $p_0$  computed from (7) should be taken.\*

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\* The domain of rationality of this investigation consists of: (1) all complex numbers, (2) the coefficients  $t_1, t_2, t_3$  of the equation  $T = 0$ , the function  $r(x)$ , and (3) the operations of addition, subtraction, multiplication, division (exclusive of the null divisor), extraction of square roots, and differentiation.

It remains still to be shown that  $\bar{y}_1, \bar{y}_2, \bar{y}_3$  form a fundamental system of  $\bar{T}(\bar{y}) = 0$ , where

$$(5') \quad y_i = q_0 \bar{y}_i + q_1 \bar{y}'_i + q_2 \bar{y}''_i \quad (i = 1, 2, 3).$$

If these functions are not linearly independent, we will have

$$c_1 \bar{y}_1 + c_2 \bar{y}_2 + c_3 \bar{y}_3 = 0;$$

from this equation and (5') it follows that

$$c_1 y_1 + c_2 y_2 + c_3 y_3 = 0,$$

a condition which is impossible, since  $y_1, y_2, y_3$  are a fundamental system of  $T(y) = 0$ .

*Note.* The propriety of replacing the equation  $T(y) = 0$  by  $\bar{T}(\bar{y}) = 0$  may be made clear from at least two points of view. I. If  $\bar{T} = 0$  is integrated then, by (5) a fundamental system of  $T = 0$  is known at once; and inversely, if  $T = 0$  is integrated, then a fundamental system of  $\bar{T} = 0$  is known by (4). II. From the standpoint of the Picard-Vessiot Theory, the equations  $T = 0$  and  $\bar{T} = 0$ , being cogredient,\* have the same Rationality Group, and consequently, the same integration theory.

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MARCH, 1911.

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\* Schlesinger, loc. cit. pp. 115 and 121.