A CLASS OF DEVELOPMENTS IN ORTHOGONAL FUNCTIONS.

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I.

1. Consider the differential equation

$$\frac{d}{dx}(k(x)y') + (\lambda^2 g(x) - h(x))y = 0; \tag{1}$$

where $k(x)$, $g(x)$ and $h(x)$ are real functions of the real variable x and where $g(x) > 0$ and $k(x) > 0$ have fourth derivatives and $h(x)$ a second derivative, limited and absolutely integrable in the Riemann sense over

$$a \leqq x \leqq b \tag{2}$$

and where moreover $g(a) = g(b)$, $g'(a) = g'(b)$, $k(a) = k(b)$ and $k'(a) = k'(b)$; subject to the conditions

$$y(a) = y(b), \qquad y'(a) = y'(b). \tag{3}$$

Denote the characteristic values for the system consisting of (1) and

$$y(a) = y(b) = 0 \tag{4}$$

by $\lambda_1^2 < \lambda_2^2 < \lambda_3^2 < \cdots$. As is well known* there exist an infinite set of values $l_1^2, l_2^2, l_3^2, \cdots$, satisfying the inequalities

$$l_1^2 < \lambda_1^2 < l_2^2 \leqq \lambda_2^2 \leqq l_3^2 < \lambda_3^2 < l_4^2 \leqq \lambda_4^2 \leqq l_5^2 < \cdots \tag{5}$$

such that when $\lambda^2 = l_j^2$, $j = 1, 2, 3, \cdots$, the system consisting of (1) and (3) is satisfied by at least one function, not identically zero. If $\lambda^2 = l_j^2 \neq l_{j-1}^2$ or l_{j+1}^2 all solutions of (1) satisfying (3) are linearly dependent. Denote by y_j a particular such solution, not identically zero. If $\lambda^2 = l_j^2 = l_{j+1}^2$ all solutions of (1) satisfy (3). Denote two particular linearly independent solutions by y_j and y_{j+1}.

If $l_j^2 \neq l_k^2$, y_j and y_k are orthogonal functions, that is they satisfy the relation

$$\int_a^b g(x)y_j(x)y_k(x)dx = 0. \tag{6}$$

* See, for example, Mason, Trans. Amer. Math. Soc., vol. VII, p. 360.

If $l_j^2 = l_{j+1}^2$ we choose y_j and y_{j+1} two solutions such that (6) holds* when k is replaced by $j + 1$.

We wish to consider the development of an arbitrary function, $f(x)$, in series of the form

$$c_1y_1 + c_2y_2 + c_3y_3 + \cdots, \tag{7}$$

where c_1, c_2, c_3, $\cdots$ are constants. If $f(x)$ is integrable from a to b the formal determination of the coefficients is immediate as in the case of Sturm-Liouville series

$$c_j = \frac{\int_a^b f(x)g(x)y_j(x)dx}{\int_a^b g(x)[y_j(x)]^2dx}. \tag{8}$$

2. In the present investigation we begin, as has generally been found necessary in such problems, by transforming† (1) to the form

$$\frac{d^2y}{dx^2} + (\lambda^2 - L(x))y = 0 \tag{9}$$

and for convenience the interval (2) into

$$0 \leqq x \leqq 2\pi. \tag{10}$$

It is deemed unnecessary to introduce new letters. Conditions (3) are transformed into

$$y(0) = y(2\pi), \qquad y'(0) = y'(2\pi), \tag{11}$$

and we denote by y_j, $j = 1, 2, \cdots$, the solution of (9) into which the old y_j is transformed. $L(x)$ has a limited integrable second derivative.

When $j \to \infty$, as is well known,‡ $\lambda_j^2 \to \infty$. Hence from (5) for large values of j, $l_j^2 > 0$. When this is the case let $l_j > 0$. Then for large values of j we can write*

$$y_j(x) = A\cos l_jx + B\sin l_jx + \frac{1}{l_j}\int_0^x L(\xi)y_j(\xi)\sin l_j(x - \xi)d\xi. \tag{12}$$

A and B are not both zero as otherwise we would have $y_j(0) = y_j'(0) = 0$ and hence $y \equiv 0$. We add arbitrarily the additional restriction,

$$A^2 + B^2 = 1. \tag{13}$$

We proceed from (12) to get an asymptotic expression for y_j as $j \to \infty$.

To begin with, $|y_j|$ does not exceed a fixed positive number inde-

* An existence proof is the same as that given—Quart. Jour. of Pure and Ap. Math., vol. XLVI, p. 9.

† See, for example, Hobson, Proc. London Math. Soc., Series 2, vol. 6, p. 375.

‡ Ibid., p. 378.

pendent of x and j. A proof of this can be given exactly as is done for the corresponding theorem for Sturm-Liouville developments* and is consequently omitted.

Since y_j satisfies (11)

$$(14)\quad A = A \cos 2\pi l_j + B \sin 2\pi l_j + \frac{1}{l_j}\int_0^{2\pi} L(\xi)u_j(\xi) \sin l_j(2\pi - \xi)d\xi,$$

$$(15)\quad B = -A \sin 2\pi l_j + B \cos 2\pi l_j + \frac{1}{l_j}\int_0^{2\pi} L(\xi)u_j(\xi) \cos l_j(2\pi - \xi)d\xi.$$

Substitute from (12) in (14) and (15) and then again in the resulting equations. α will here, as in the remainder of this paper, denote a number independent of x which does not exceed in absolute value a positive number independent of j. $\alpha(x)$ will denote a function which never exceeds a fixed positive number independent of x and j. *α will generally denote quite different functions when used in different places*, not a particular function as would naturally be assumed. In addition let

$$\int_0^{2\pi} L(\xi)d\xi = 2N, \qquad l_j\int_0^{2\pi} L(\xi)\cos 2l_j\xi d\xi = 2K,$$

$$l_j\int_0^{2\pi} L(\xi)\sin 2l_j\xi d\xi = 2M \qquad \text{and} \qquad \tan l_j\pi = z.$$

We get after simplification

$$(16)\quad \begin{aligned} &A\left(2l_j^2z^2 - 2l_jNz - 2Kz + M - Mz^2 + \frac{N^2}{2} - \frac{N^2}{2}z^2\right) \\ &\qquad = B(2l_j^2z + 2Mz - Nl_j - N^2z + Nl_jz^2 + K - Kz^2) + \frac{\alpha}{l_j}. \end{aligned}$$

$$(17)\quad \begin{aligned} &A(2l_j^2z - Nl_j + Nl_jz^2 - K + Kz^2 - 2Mz - N^2z) \\ &\qquad = B\left(-2l_j^2z^2 + M - Mz^2 + 2Nl_jz - \frac{N^2}{2} + \frac{N^2}{2}z^2 - 2Kz\right) + \frac{\alpha}{l_j}. \end{aligned}$$

Suppose that as $j \to \infty$ l_jz does not remain finite then from (16) and (17)

$$2A + \beta_1 = \frac{2}{z}B + \beta_2, \qquad \frac{2}{z}A + \beta_3 = -2B + \beta_4,$$

where β_1, β_2, β_3 and β_4 are infinitesimal on some special set of values of λ or are zero. But these equations are impossible for values of A and B other than $A = B = 0$. Consequently l_jz remains finite as $j \to \infty$. The

* Ibid., p. 376.

elimination of A and B is now simple and yields the equation

$$\begin{aligned}(18)\qquad &\left[4l_j^4 - N^2l_j^2 - M^2 - K^2 + \frac{N^4}{4}\right]z^4 \\ &- 4l_j^3Nz^3 + \left[4l_j^4 - 2M^2 - 2K^2 + \frac{N^4}{2}\right]z^2 \\ &- 4l_j^3Nz + \left[N^2l_j^2 - K^2 - M^2 + \frac{N^4}{4}\right] = \frac{\alpha}{l_j}.\end{aligned}$$

The left-hand member of this equation is devisable by $(1 + z^2)$. Consequently for the real values of z, in which alone we are interested,

$$(19)\qquad z = \frac{2l_j^3N \pm \sqrt{4(M^2+K^2)l_j^4 - \left(\frac{N^4}{4} - (M^2+K^2)\right)^2}}{4l_j^4 - N^2l_j^2 - (M^2+K^2) + \frac{N^4}{4}} + \frac{\alpha}{l_j^3}.$$

But $\lambda_{2n-1} \rightarrow n - \frac{1}{2}$ and $\lambda_{2n+1} \rightarrow n + \frac{1}{2}$ as $n \rightarrow \infty$.* Consequently from (5) and (19) $l_{2n} = n + (\alpha/n)$ and $l_{2n+1} = n + (\alpha/n)$ and consequently $K = (\alpha/n)$ and then

$$(20)\qquad \begin{cases} \tan \pi l_{2n+1} = \dfrac{N}{2n} + \dfrac{M}{2n^2} + \dfrac{\alpha}{n^3}, \\ \tan \pi l_{2n} = \dfrac{N}{2n} - \dfrac{M}{2n^2} + \dfrac{\alpha}{n^3}, \end{cases}$$

and consequently

$$l_{2n} = n + \frac{N}{2n\pi} + \frac{\alpha}{n^2},$$

$$l_{2n+1} = n + \frac{N}{2n\pi} + \frac{\alpha}{n^2}.$$

Suppose $M \nrightarrow 0$ which is the case if $L(2\pi) \neq L(0)$. The substitution of these values in either (16) or (17) yields, giving a positive value to A which is allowable, $A = (1/\sqrt{2}) + (\alpha/n)$, $B = \pm(1/\sqrt{2}) + (\alpha/n)$. Repeated substitution in (12) yields

$$(21)\qquad \begin{cases} u_{2n} = \dfrac{1}{\sqrt{2}}\cos l_{2n}x + \dfrac{1}{\sqrt{2}}\sin l_{2n}x + \alpha\dfrac{\cos l_{2n}x}{n} + \alpha\dfrac{\sin l_{2n}x}{n} + \dfrac{\alpha(x)}{n^2},\dagger \\ u_{2n+1} = \dfrac{1}{\sqrt{2}}\cos l_{2n+1}x - \dfrac{1}{\sqrt{2}}\sin l_{2n+1}x + \alpha\dfrac{\cos l_{2n+1}x}{n} \\ \qquad\qquad + \alpha\dfrac{\sin l_{2n+1}x}{n} + \dfrac{\alpha(x)}{n^2}. \end{cases}$$

* Ibid., p. 378.

† With the values obtained for A and B it is readily shown that

$$\int_0^x L(\xi)u_j(\xi)\sin l_j(x-\xi)d\xi = \frac{\alpha(x)}{l}.$$

In case $M \to 0$ a supplementary discussion is necessary. This discussion will be of a general character; in it further properties of the α's in (21) will be developed. Denote $\cos 2\pi l_j$ by c, $\sin 2\pi l_j$ by s, $\cos l_j \xi_i$ by c_i, $\sin l_j \xi_i$ by s_i, $\cos 2l_j\xi_i$ by C_i and $\sin 2l_j\xi_i$ by S_i, $i = 1, 2, 3, \cdots$. Let

$$\varphi_i = (sc_1 - cs_1)(s_1c_2 - c_1s_2)\cdots(s_{i-1}c_i - c_{i-1}s_i)(Ac_i + Bs_i),$$
$$\chi_i = (cc_1 + ss_1)(s_1c_2 - c_1s_2)\cdots(s_{i-1}c_i - c_{i-1}s_i)(Ac_i + Bs_i).$$

Substitution in (14) and (15) from (12) repeatedly yields

$$\begin{aligned} A = {} & Ac + Bs + \frac{1}{l_j}\int_0^{2\pi} L(\xi_1)\varphi_1 d\xi_1 + \frac{1}{l_j^2}\int_0^{2\pi}\int_0^{\xi} L(\xi_1)L(\xi_2)\varphi_2 d\xi_2 d\xi_1 \\ & + \cdots + \frac{1}{l_j^p}\int_0^{2\pi}\int_0^{\xi_1}\cdots\int_0^{\xi_{p-1}} L(\xi_1)L(\xi_2) \\ & \qquad \cdots L(\xi_p)\varphi_p d\xi_p d\xi_{p-1}\cdots d\xi_1 + \cdots, \end{aligned} \tag{22}$$

$$\begin{aligned} B = {} & -As + Bc + \frac{1}{l_j}\int_0^{2\pi} L(\xi_1)\chi_1 d\xi_1 \\ & + \frac{1}{l_j^2}\int_0^{2\pi}\int_0^{\xi_1} L(\xi_1)L(\xi_2)\chi_2 d\xi_2 d\xi_1 + \cdots + \frac{1}{l_j^p}\int_0^{2\pi}\int_0^{\xi_1} \\ & \cdots \int_0^{\xi_{p-1}} L(\xi_1)L(\xi_2)\cdots L(\xi_p)\chi_p d\xi_p d\xi_{p-1}\cdots d\xi_1 + \cdots. \end{aligned} \tag{23}$$

Consider now the expression φ_i, omitting the first and last factors, we have $(s_1c_2 - c_1s_2)\cdots(s_{i-1}c_i - c_{i-1}s_i)$ which we denote by ψ_i. If i is odd, $\psi_i c_1 c_i$ is the same as $\psi_i s_1 s_i$ except that every s_k^2, $k = 1, \cdots, i$, is replaced by c_k^2 and vice versa. If i is even there is this same relation but for a change of sign. Replace now in $\psi_i c_1 c_i$ and in $\psi_i s_1 s_i$ each s_k^2 by $(1 - C_k)/2$, each c_k^2 by $(1 + C_k)/2$ and each $c_k s_k$ by $\frac{1}{2}S_k$. Multiply out all parentheses. $\psi_i c_1 c_i$ and $\psi_i s_1 s_i$ now differ only in the signs of some of the terms, namely, those containing an odd number of C_k's. The same relation exists between $c_i s_1 \psi_i$ and $s_i c_1 \psi_i$.

Now in (22) and (23) collect all the coefficients of As, Bs, Ac and Bc. We have absolutely convergent infinite series, as is seen by comparison with the series

$$\sum_{p=1}^{\infty}\frac{1}{p!}\left(\frac{2}{l_j}\int_0^{2\pi} |L(x)|\,dx\right)^p.$$

Let

$$\frac{s}{\cos^2 l_j\pi} = 2z \qquad \text{and} \qquad \frac{c}{\cos^2 l_j\pi} = 1 - z^2$$

as previously. The notation for the coefficients will be obvious. We get

$$(24)\quad \begin{cases} A(2l_j^2z^2 - 2l_j\bar{N}z - 2\bar{K}z + \bar{M} - \bar{M}z^2 + P - Pz^2) \\ \qquad = B(2l_j^2z - \bar{N}l_j + 2\bar{M}z - 2Pz + \bar{N}lz^2 + \bar{K} - \bar{K}z^2), \\ A(2l_j^2z - \bar{N}l_j + \bar{N}l_jz^2 - \bar{K} + \bar{K}z^2 - 2\bar{M}z - 2Pz) \\ \qquad = B(-2l_j^2z^2 + \bar{M} - \bar{M}z^2 + 2\bar{N}l_jz - P + \bar{P}z^2 - 2\bar{K}z). \end{cases}$$

The elimination of A and B and the solution of the consequent equation as in the case already discussed where $M \not\to 0$ yields

$$(25)\quad z = \frac{4l_j^3\bar{N} \pm \sqrt{l_j^4(16\bar{M}^2 + 16\bar{K}^2 - 16P^2 + 16N^2P - 4N^4) + 4l_j^2(\bar{M}^2 + \bar{K}^2 - P^2)(2\bar{N}^2 - 4P) - 4(P^2 - \bar{M}^2 - \bar{K}^2)}}{8l_j^4 - 8l_j^2P + 4\bar{N}^2l_j^2 + 4P^2 - 4\bar{M}^2 - 4\bar{K}^2}.$$

Let us assume that $L(x)$ is analytic over* (10), *an assumption not made when $M \not\to 0$.* In the latter case it was assumed only that $L^{\mathrm{IV}}(x)$ was limited and absolutely integrable.

Let

$${}_qE_p{}^{(n)} = \int_0^{2\pi}\int_0^{\xi_1}\cdots\int_0^{\xi_{n-1}} L(\xi_1)L(\xi_2)\cdots L(\xi_n)c_{j_1}\cdots c_{j_p}s_{k_1}\cdots s_{k_q}d\xi_1 d\xi_2\cdots d\xi_q,$$

where $j_1, \cdots, j_p, k_1, \cdots, k_q$ are distinct numbers of the succession $1, \cdots, n$ and $p + q \leqq n$. By integration by parts it is readily shown that ${}_qE_p{}^{(n)}$ is expansible in the form

$${}_qE_p{}^{(n)} = \frac{E_1}{l_j{}^{q+p}} + \frac{E_2}{l_j{}^{q+p+1}} + \cdots,$$

where $E_1, E_2, \cdots$, are polynomials in s and c.

Now write from (25), as is immediate,

$$z = \frac{z_1}{l_j} + \frac{\alpha}{l_j^2},$$

where z_1 is a constant. This result can be used to determine s and c of the form

$$s = \frac{s_1}{l_j} + \frac{\alpha}{l_j^2}, \qquad c = 1 + \frac{c_1}{l_j} + \frac{\alpha}{l_j^2},$$

where s_1 and c_1 are constants. Consequently

$$\bar{M} = \frac{M_1}{l_j} + \frac{\alpha}{l_j^2}, \qquad \bar{K} = \frac{K_1}{l_j} + \frac{\alpha}{l_j^2}, \qquad \bar{N} = N + \frac{N_1}{l_j} + \frac{\alpha}{l_j^2},$$

$$P = \frac{N^2}{2} + \frac{P_1}{l_j} + \frac{\alpha}{l_j^2},$$

* It will appear that this is more than is generally necessary.

where M_1, K_1, N_1 and P_1 are constants. These values in (25) determine

$$z = \frac{z_1}{l_j} + \frac{z_2}{l_j^2} + \frac{\alpha}{l_j^3},$$

from which in a similar manner

$$\overline{M} = \frac{M_1}{l_j} + \frac{M_2}{l_j^2} + \frac{\alpha}{l_j^3},$$

where M_2 is also a constant; similarly for $\overline{K}$, $\overline{N}$ and P. These values in (25) then determine

$$z = \frac{z_1}{l_j} + \frac{z_2}{l_j^2} + \frac{z_3}{l_j^3} + \frac{\alpha}{l_j^4}, \qquad \text{etc.}$$

The process can be carried on as far as is desired. Suppose that at each stage of this process we substitute in the first of (24) and together with $A^2 + B^2 = 1$ solve for A and B, either A and B are indeterminate or are in the form, taking the positive sign with the radical

$$A = A_1 + \frac{a_1}{l_j} + \frac{\alpha}{l_j^2}, \qquad B = B_1 + \frac{b_1}{l_j} + \frac{\alpha}{l_j^2},$$

taking the negative sign

$$A = A_2 + \frac{a_2}{l_j} + \frac{\alpha}{l_j^2}, \qquad B = B_2 + \frac{b_2}{l_j} + \frac{\alpha}{l_j^2}.$$

A_1, A_2, B_1, B_2, a_1, a_2, b_1 and b_2 are fixed real numbers. If no matter how far the process just described is carried A and B remain indeterminate, equations (11) place no restriction on the solution and l_j is a double value. As previously

$$l_{2n} = n + \frac{N}{2n\pi} + \frac{\alpha}{n^2}, \qquad l_{2n+1} = n + \frac{N}{2n\pi} + \frac{\alpha}{n^2}.$$

We consequently can write

$$(26)\quad \begin{cases} y_{2n+1} = A_1 \cos l_{2n+1}x + B_1 \sin l_{2n+1}x \\ \qquad + \dfrac{a_1 - B_1N(x)}{n} \cos l_{2n+1}x + \dfrac{b_1 + A_1N(x)}{n} \sin l_{2n+1}x + \dfrac{\alpha(x)}{n^2}, \\ y_{2n} = A_2 \cos l_{2n}x + B_2 \sin l_{2n}x \\ \qquad + \dfrac{a_2 - B_2N(x)}{n} \cos l_{2n}x + \dfrac{b_2 + A_2N(x)}{n} \sin l_{2n}x + \dfrac{\alpha(x)}{n^2}. \end{cases}$$

which are the desired asymptotic forms, $2N(x) = \int_0^x L(x)d(x) \cdot y_{2n}$ and

y_{2n+1} are orthogonal, which fact together with (13) necessitates

$$(27)\quad \begin{cases} A_i^2 + B_i^2 = 1, \ (i = 1, 2), \\ A_1A_2 + B_1B_2 = A_1B_1 + A_2B_2 = 0, \quad A_1^2 + A_2^2 = B_1^2 + B_2^2 = 1, \\ A_1a_1 + B_1b_1 = A_2a_2 + B_2b_2 = A_1a_1 + A_2a_2 = B_1b_1 + B_2b_2 = 0, \\ A_1a_2 + A_2a_1 + B_1b_2 + B_2b_1 = 0, \\ A_1b_1 + B_1a_1 + A_2b_2 + B_2a_2 = 0. \end{cases}$$

3. Adopt the notation of (26) even in case $M \not\to 0$, for then simply

$$A_1 = A_2 = \frac{1}{\sqrt{2}}, \qquad B_1 = -B_2 = \frac{1}{\sqrt{2}}.$$

Let $v_{2n+1} = A_1 \cos nx + B_1 \sin nx$, $v_{2n} = A_2 \cos nx + B_2 \sin nx$. Let $f(x)$ be an integrable function and let

$$\gamma_{2n+1} = \frac{\int_0^{2\pi} f(x) v_{2n+1} dx}{\int_0^{2\pi} (v_{2n+1})^2 dx}, \qquad \gamma_{2n} = \frac{\int_0^{2\pi} f(x) v_{2n} dx}{\int_0^{2\pi} (v_{2n})^2 dx}$$

It can immediately be verified by means of (27) that

$$\gamma_{2n}v_{2n} + \gamma_{2n+1}v_{2n+1} = a_n \cos nx + b_n \sin nx,$$

where a_n and b_n are the Fourier constants for $f(x)$. Moreover, replacing l_{2n+1} and l_{2n} by the values obtained for them in terms of n and letting

$$c_j = \frac{\int_0^{2\pi} f(x) y_j(x) dx}{\int_0^{2\pi} [y_j(x)]^2 dx}$$

from 26

$$(28)\qquad c_{2n}y_{2n} + c_{2n+1}y_{2n+1} = a_n \cos nx + b_n \sin nx + p_n,$$

where by (27)

$$p_n = \alpha \frac{a_n}{n^2} + \alpha \frac{b_n}{n^2} + \frac{\alpha}{n^2} \int_0^{2\pi} f(\xi) \alpha(\xi) d\xi.$$

In case that A_1, B_1, A_2 and B_2 are arbitrary but for (13), that is, where $l_j = l_{2n} = l_{2n+1}$, a double value, we choose for definiteness $A_1 = B_2 = 1$, $A_2 = B_1 = 0$ and as a result still have (28).

Let

$$s_{2n} = \sum_{n=1}^{n} c_{2n}y_{2n} + c_{2n+1}y_{2n+1}$$

and σ_{2n} the sum of the first $2n - 1$ terms of the Fourier series for $f(x)$. The series $p_0 + p_1 + \cdots$ converges uniformly over $0 \leqq x \leqq 2\pi$. *Then* $s_{2n} - \sigma_{2n}$ *converges uniformly in* x *over* $0 \leqq x \leqq 2\pi$.

Assume first that $f(x)$ is analytic when $0 \leqq x \leqq 2\pi$. Both s_n and σ_n converge to $f(x)$ when $0 < x < 2\pi$.* But $s_{2n} - \sigma_{2n}$ is continuous and converges uniformly. Consequently $s_{2n} - \sigma_{2n}$ converges uniformly to zero over the closed interval $0 \leqq x \leqq 2\pi$.

Now let

$$s_{2n}(x) - \sigma_{2n}(x) = \int_0^{2\pi} f(\xi)\Phi_n(x, \xi)d\xi,$$

$$\Phi_n(x, \xi) = \frac{\alpha(x, \xi)}{n^2},$$

where $\alpha(x, \xi)$ for all values of n remains in absolute value less than a fixed positive number independent of x and ξ. It results that $\Phi_n(x, \xi)$ remains in absolute value less than a fixed number independent of x, n and ξ. No assumptions as to the character of $f(x)$ other than integrability are made here.

Next let $f(x)$ be continuous, $0 \leqq x \leqq 2\pi$, but not necessarily analytic. Form a sequence of analytic functions, $f_1, f_2, f_3, \cdots$, approaching f uniformly, $0 \leqq x \leqq 2\pi$. Obviously

$$\int_0^{2\pi} f(\xi)\Phi_n(x, \xi)d\xi = \int_0^{2\pi} (f(\xi) - f_n(\xi))\Phi_m(x, \xi)d\xi + \int_0^{2\pi} f_n(\xi)\Phi_m(x, \xi)d\xi.$$

Let η be arbitrarily small. Since Φ_n is bounded we can choose an $\bar{n}$ such that when $n \geqq \bar{n}$ the first integral is in absolute value less than $\frac{1}{2}\eta$. Having chosen $\bar{n}$ choose $\bar{m}$ so that when $m \geqq \bar{m}$ the last integral is in absolute value less than $\frac{1}{2}\eta$. Consequently $|s_{2n} - \sigma_{2n}| < \eta$. Consequently $s_{2n} - \sigma_{2n}$ converges to zero uniformly over the closed interval $0 \leqq x \leqq 2\pi$.

Extension to discontinuous functions is made by setting up a sequence of continuous functions, $f_1, f_2, \cdots$, such that

$$\int_0^{2\pi} |f_n(\xi) - f(\xi)|d\xi \to 0$$

and reasoning much as before.† *The result is obtained that* $s_{2n} - \sigma_{2n} \to 0$ *uniformly when* $0 \leqq x \leqq 2\pi$.

The following theorem is readily concluded.

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* A. C. Dixon, Proc. L. Math. Soc., 1905, p. 99.

† See Haar, Math. Ann., Bd. 69, S. 355, where similar reasoning is carried through for the Sturm-Liouville and cosine series. See also Hobson, Theory of Functions of a Real Variable, p. 532, for classification of functions defined by sequences.

If $f(x)$ is a function integrable in the Lebesgue sense from 0 to 2π. The development $\sum_{\infty}^{n=1}(c_{2n}y_{2n}+c_{2n+1}y_{2n+1})$ converges at any particular point of $0 \leqq x \leqq 2\pi$ when and only when the Fourier's series for $f(x)$ converges at that point and to the same value. It converges uniformly over the whole interval or any subinterval when and only when this is true of the Fourier's series and diverges to ∞ or $-\infty$ when and only when the Fourier's series does. Moreover it is summable by the method of the arithmetic mean when and only when the same thing is true of the Fourier's series and to the same value.

It is to be noted that results from this theorem are immediately transferable to series (7) over the interval (2).*

II.

4. Very similar to the problem just discussed in I is that of developing a function in terms of the infinite succession of solutions of (9) satisfying

$$y(0) = -y(2\pi), \qquad y'(0) = -y'(2\pi). \tag{29}$$

The work throughout is the same as that gone through in I except for the details of the algebra which fact apparently makes a proof by the method of this paper to cover both cases impractical. The results will, however, be readily surmised and I shall give them without anything resembling a detailed proof.

Let the characteristic values be denoted by $l_j'^2$, $j = 1, 2, 3, \cdots$, and for large values of j let $z = \cot l_j'\pi$. Analogous to equations (24) we have

$$A(2l_j'^2z^2 + 2l_j'\overline{N}'z + 2\overline{K}'z - \overline{M}'z^2 + \overline{M}' - P'z^2 + P')$$
$$= B(-2l_j'^2z - 2\overline{M}'z - \overline{K}'z^2 + \overline{K}' + \overline{N}'l_jz^2 - \overline{N}'l_j + 2P'z),$$
$$A(2l_j'^2z - \overline{N}'l_j'z^2 + \overline{N}'l_j' - 2\overline{M}'z - \overline{K}'z^2 + \overline{K}' - 2P'z)$$
$$= B(2l_j'^2z^2 - 2\overline{K}'z + \overline{M}'z^2 - \overline{M}' + 2\overline{N}'l_jz - P'z^2 + P'),$$

where, in case $L(0) \neq L(2\pi)$, P' is to be replaced by $N^2/2$, the dashes dropped and α/n added as in (16), (17). These equations are solved for z and together with $A^2 + B^2 = 1$ for A and B as in I. It is found that $z \to 0$. Hence from the inequalities

$$l_1'^2 \leqq \lambda_1^2 \leqq l_2'^2 < \lambda_2^2 < l_3'^2 \leqq \lambda_3^2 \leqq l_4'^2 < \cdots,$$

$$l_{2n-1}' = \frac{2n-1}{2} + \frac{\alpha}{n} \qquad \text{and} \qquad l_{2n} = \frac{2n+1}{2} + \frac{\alpha}{n}.$$

* See Hobson, Proceeding L. Math. Soc., Series 2, vol. VI, p. 387, where the same thing is done for Sturm-Liouville series.

Other work is a close parallel to that already carried through with the result that:

The development for $f(x)$ *in terms of solutions of* (9) *satisfying* (29) *bears the same relation to the development for* $f(x)$ *in terms of* $\sin [(2n + 1)/2]x$ *and* $\cos [(2n + 1)/2]x$, $n = 0, 1, 2, \cdots$, *as is borne by the series discussed in I to the Fourier's series for* $f(x)$.

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